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Tensorial products of functional ARMA processes

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ABSTRACT

We study the structure of tensorial products for the autoregressive and moving average processes (X_n), with values in a Hilbert space H and with innovations that are martingale differences.

The obtained models are $ARMA(H \otimes H)$ processes, possibly non standard. We provide criteria for the standardness of these models, we specify the results in the real case, give some examples and consider some applications.

1. Introduction

Linear processes with values in a Hilbert space H appear in high dimensional statistics (cf. Grenander [1], Bosq [2], Ferraty and Vieu [3] among others). In particular, they allow one to construct linear representations of various continuous time processes. A typical and simple example is the Ornstein–Uhlenbeck process (see below).

A general definition of linear processes on H has the form

$$X_n = \varepsilon_n + \sum_{j \geq 1} \ell_j(\varepsilon_{n-j}), \quad n \in \mathbb{Z},$$

where $(\varepsilon_n, n \in \mathbb{Z})$ is a H-white noise and $(\ell_j, j \geq 1)$ is a sequence of linear operators on H, possibly not continuous. This general form appears as more convenient than the classical one (where the ℓ_j's are supposed to be continuous) because it is the natural Wold decomposition of (X_n) (cf. Bosq [4,5]).

Now, a linear process is said to be **standard** if the ℓ_j's are continuous. This is of course a desirable property since, if ℓ_j is not continuous, one cannot construct an estimator of ℓ_j which converges in the operator's norm sense.

In the current paper, we study the quadratic transforms of linear processes on H. In the real case, these transforms have been considered by Granger and Newhold [6], Phillips and Solo [7] and more recently Choi and Taniguchi [8], among others.

In our Hilbertian context, "product" is replaced by "tensorial product" : we study the structure of processes of the form $(X_{n-k} \otimes X_n, n \in \mathbb{Z}, (k \geq 0))$. Since the problem is rather intricate we will only consider the case where (X_n) is a standard autoregressive process of order 1 ($ARH(1)$) or a standard moving average of order 1 ($MAH(1)$). These models have been used in practice (cf. Bosq [2], Bosq and Blanke [9]).

The main results may be summarized as follows : if (X_n) is an $(ARH(1))$ or a $(MAH(1))$, then $(X_{n-k} \otimes X_n)$ is an $ARMA(H \otimes H)$ process, possibly non standard; moreover, under some regularity conditions, $(X_{n-k} \otimes X_n)$ is standard or has a standard part.

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These results allow one to obtain consistent estimators of the autocovariance of (X_n) and to estimate the linear operators associated with the structure of $(X_{n-k} \otimes X_n)$.

In the next section, we introduce some notation and assumptions. Section 3 is devoted to tensorial products of $ARH(1)$ processes when Section 4 deals with tensorial products of $MAH(1)$ processes. In Section 5 we study standardness of tensorial products for the special case of a gaussian $MAH(1)$. Applications to estimation are briefly indicated in Section 6. Finally, the proofs and some preliminary useful results appear in Section 7.

2. Notation and assumptions

Let H be a separable real Hilbert space equipped with its norm $\|.\|$ and its scalar product $\langle\cdot,\cdot\rangle$. Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $L^2_H(P) = L^2_H(\Omega, \mathcal{A}, P)$ be the Hilbert space of (classes of) random variables X, H-valued and such that $\mathbb{E}\|X\|^2 < \infty$. If X and Y are in $L^2_H(P)$ and are zero-mean, the cross covariance operators of X and Y are defined as

$$C_{X,Y}(x) = \mathbb{E}(\langle X, x\rangle Y), \quad x \in H$$

and

$$C_{Y,X} = C^*_{X,Y}$$

where $*$ denotes "adjoint". The covariance operator of X is defined as

$$C_X = C_{X,X}.$$

The tensorial product $a \otimes b$ is the operator defined by

$$(a \otimes b)(x) = \langle a, x\rangle b, \quad x \in H \ (a, b \in H).$$

$\mathcal{S} = H \otimes H$ denotes the Hilbert space of Hilbert Schmidt operators on H (cf. Akhiezer and Glazman [10] or Bosq [2]).

Let $\varepsilon = (\varepsilon_n, n \in \mathbb{Z})$ be a sequence of H-valued random variables, we denote by B_n the σ-algebra generated by $\{\varepsilon_t, t \le n\}$ and by $\mathbb{E}^{\mathcal{B}_n}$ the conditional expectation with respect to $\mathcal{B}_n$. We make the following assumptions:

$$\mathbf{A}_1 - \mathbb{E}\|\varepsilon_n\|^4 < \infty, \qquad \mathbb{E}(\varepsilon_n^{\otimes 4}) = \mathbb{E}(\varepsilon_0^{\otimes 4}), \qquad \mathbb{E}^{\mathcal{B}_{n-1}}(\varepsilon_n^{\otimes 3}) = 0,$$

$$\mathbb{E}^{\mathcal{B}_{n-1}}(\varepsilon_n^{\otimes 2}) = C_{\varepsilon_0} \neq 0, \qquad \mathbb{E}^{\mathcal{B}_{n-1}}(\varepsilon_n) = 0; \quad n \in \mathbb{Z}.$$

If A_1 holds, ε is a H-white noise and a martingale difference, in particular

$$C_{\varepsilon_n,\varepsilon_m} = 0, \quad n \neq m.$$

Note that, here, a "martingale difference" is used in an extended sense since the family $(\mathcal{B}_n)$ is indexed by $\mathbb{Z}$ instead of $\mathbb{N}$.

Example 1. A strong white noise such that $\mathbb{E}\|\varepsilon_0\|^4 < \infty$ and $\mathbb{E}^{\mathcal{B}_{n-1}}(\varepsilon_n^{\otimes 3}) = 0$ satisfies A_1. In particular a gaussian white noise satisfies A_1.

Example 2. Let $U = (U_n, n \in \mathbb{Z})$ be a sequence of H-valued i.i.d. r.v.'s such that $0 < \mathbb{E}\|U_n\|^4 < \infty$, $\mathbb{E}(U_n^{\otimes 3}) = 0$, $\mathbb{E}(U_n) = 0$ and let V be a real r.v. such that $P(V = -v) = P(V = v) = 1/2$, $(v \neq 0)$. Then, if $U \perp V$, $\varepsilon_n = U_n V$, $n \in \mathbb{Z}$ satisfies A_1.

Stationary processes. Let $X = (X_n, n \in \mathbb{Z})$ be a H-valued zero mean, stationary regular process (cf. Bosq and Blanke [9]) and let $\varepsilon = (\varepsilon_n, n \in \mathbb{Z})$ be the innovation of X.

X is a H-moving average of order 1 ($MAH(1)$) if

$$X_n = \varepsilon_n + \lambda(\varepsilon_{n-1}), \quad n \in \mathbb{Z} \tag{2.1}$$

where $\lambda(\varepsilon_{n-1})$ denotes the orthogonal projection of X_n on

$$g_{\varepsilon_{n-1}} = \overline{sp}\{\ell(\varepsilon_{n-1}),\ \ell \in \mathcal{L}\}$$

in $L^2_H(P)$, $\mathcal{L} = \mathcal{L}(H, H)$ is the space of continuous linear operators from H to H, and the closure is taken in $L^2_H(P)$. $\mathcal{L}$ is equipped with its norm: $\|\ell\|_{\mathcal{L}} = \sup_{\|x\|\le 1} \|\ell(x)\|$.

Similarly X is a H-autoregressive process of order 1 ($ARH(1)$) if

$$X_n = \mu(X_{n-1}) + \varepsilon_n, \quad n \in \mathbb{Z} \tag{2.2}$$

where $\mu(X_{n-1})$ is the orthogonal projection of X_n on

$$g_{X_{n-1}} = \overline{sp}\{\ell(X_{n-1}), \ell \in \mathcal{L}\}.$$

The above definitions appear in Bosq [4,5]. The "linearly closed" spaces $g_{\varepsilon_{n-1}}$ and $g_{X_{n-1}}$ appear in [11].

Now the $MAH(1)X$ is said to be **standard** if there exists $a \in \mathcal{L}$ such that $\|a^{j_0}\|_{\mathcal{L}} < 1$ for some $j_0 \ge 1$ and $\lambda(\varepsilon_{n-1}) = a(\varepsilon_{n-1})$. Similarly the ARH(1) X is standard if there exists $\rho \in \mathcal{L}$ with $\|\rho^{j_0}\|_{\mathcal{L}} < 1$ for some $j_0 \ge 1$ and $\mu(X_{n-1}) = \rho(X_{n-1})$. A condition for standardness is studied in Bosq [4,5].

Finally note that a standard ARH(1) process admits the decomposition

$$X_n = \sum_{j=0}^{\infty} \rho^j(\varepsilon_{n-j}), \quad n \in \mathbb{Z} \tag{2.3}$$

where the series converges in $L^2_H(P)$ and almost surely.

In the following "$n \in \mathbb{Z}$" will be omitted except if there is some ambiguity.

3. Tensorial products of ARH(1) processes

Consider a standard ARH(1) process defined by

$$X_n = \rho(X_{n-1}) + \varepsilon_n \tag{3.1}$$

where $\|\rho^{j_0}\|_{\mathcal{L}} < 1$ for some $j_0 \geq 1$ and (ε_n) is a H-white noise.

The following statement gives the structure of the $\mathcal{S}$-valued process

$$Z_n^{(0)} = X_n \otimes X_n - C_{X_0}. \tag{3.2}$$

Proposition 3.1. *If A_1 holds $(Z_n^{(0)})$ is a standard AR$\mathcal{S}$(1):*

$$Z_n^{(0)} = R(Z_{n-1}^{(0)}) + E_n \tag{3.3}$$

where R is defined by

$$R(s) = \rho s \rho^*, \quad s \in \mathcal{S} \tag{3.4}$$

and

$$E_n = (X_{n-1} \otimes \varepsilon_n)\rho^* + \rho(\varepsilon_n \otimes X_{n-1}) + \varepsilon_n \otimes \varepsilon_n - C_{\varepsilon_0}. \tag{3.5}$$

It follows that

$$\|R^j\|_{\mathcal{L}(\mathcal{S},\mathcal{S})} \leq \|\rho^j\|^2_{\mathcal{L}(H,H)}, \quad j \geq 1 \tag{3.6}$$

and that (E_n) is the innovation of $(Z_n^{(0)})$ and a martingale difference with respect to $(\mathcal{B}_n)$.

Note that, if ρ is symmetric compact, i.e.

$$\rho = \sum_{j\geq 1} \rho_j(e_j \otimes e_j), \quad \text{where } (\rho_j) \to 0$$

then R is also symmetric compact:

$$R = \sum_{j,j'\geq 1} \rho_j \rho_{j'} (e_j \otimes e_j) \otimes (e_{j'} \otimes e_{j'}).$$

Proposition 3.1 is an extension of Lemma 4.1 p. 96 in [2].

Example 3 (*The Real Case*). If $X_n = \rho X_{n-1} + \varepsilon_n \quad (|\rho| < 1)$ where (ε_n) satisfies A_1 then

$$X_n^2 - \frac{\sigma^2}{1-\rho^2} = \rho^2\left(X_{n-1}^2 - \frac{\sigma^2}{1-\rho^2}\right) + E_n$$

then $R = \rho^2$ and

$$E_n = \varepsilon_n^2 - \sigma^2 + 2\rho\varepsilon_n X_{n-1}. \quad \blacksquare$$

Example 4 (*Ornstein–Uhlenbeck Process*). Consider an Ornstein–Uhlenbeck process defined as

$$\xi_t = \int_{-\infty}^{t} \mathrm{e}^{-\theta(t-s)} \mathrm{d}W(s), \quad t \in \mathbb{R} \quad (\theta > 0) \tag{3.7}$$

where W is a bilateral Wiener process with parameter σ^2. Putting

$$X_n(t) = \xi_{n+t}, \quad 0 \leq t \leq 1;\ n \in \mathbb{Z} \tag{3.8}$$

one obtains a ARH(1) process ([2]) where

$$H = L^2([0,1], \mathcal{B}_{[0,1]}, \lambda + \delta_{(1)})$$

with λ the Lebesgue measure and $\delta_{(1)}$ the Dirac measure at 1, and where

$$\rho = \mathbb{1}_{\{1\}} \otimes e_\theta \quad (e_\theta(t) = \mathrm{e}^{-\theta t}),$$

the associated white noise is gaussian:

$$\varepsilon_n(t) = \sigma \int_n^{n+t} \mathrm{e}^{-\theta(n+t-s)} \mathrm{d}W(s), \quad 0 \leq t \leq 1; n \in \mathbb{Z},$$

consequently A_1 holds and Proposition 3.1 gives:

$$Z_n^{(0)} = R(Z_{n-1}^{(0)}) + E_n \tag{3.9}$$

with

$$R(s) = (\mathbb{1}_{\{1\}} \otimes e_\theta)s(e_\theta \otimes \mathbb{1}_{\{1\}}), \quad s \in \mathcal{S}$$

and where E_n, defined as an element of $H \otimes H$, is given by

$$E_n(s,t) = \left[\varepsilon_n(s)\varepsilon_n(t) - \frac{\sigma^2}{2\theta}\left(\mathrm{e}^{-\theta|t-s|} - \mathrm{e}^{-\theta(s+t)}\right)\right] + X_{n-1}(1)\left(\mathrm{e}^{-\theta s}\varepsilon_n(t) + \mathrm{e}^{-\theta t}\varepsilon_n(s)\right); \quad 0 \le s, t \le 1. \quad \blacksquare \tag{3.10}$$

Note that this example can be extended by replacing W with a zero-mean process with independent and stationary increments, for example a compensated homogenous Poisson process.

Other extensions may be obtained by considering "linear" stationary diffusion processes.

We now consider the processes

$$Z_n^{(k)} = X_{n-k} \otimes X_n - \rho^k C_{X_0}, \quad n \in \mathbb{Z}\ (k \ge 1)$$

where (X_n) satisfies (3.1).

The structure of $(Z_n^{(k)})$ appears in the next proposition:

Proposition 3.2. *If A_1 holds then, for each $k \ge 1$, $(Z_n^{(k)})$ is a ARMA $\mathcal{S}$ (1,k) process with a standard autoregressive part:*

$$Z_n^{(k)} - R(Z_{n-1}^{(k)}) = E_n + \sum_{j=1}^{k} \lambda_j(E_{n-j}), \quad n \in \mathbb{Z} \tag{3.11}$$

where R is defined in (3.4), (E_n) is the innovation of $(Z_n^{(k)})$ and $\sum_{j=1}^{k} \lambda_j(E_{n-j})$ is the orthogonal projection of $Z_n^{(k)} - R(Z_{n-1}^{(k)})$ on $\overline{sp}\{\sum_{j=1}^{k} \ell_j(E_{n-j});\ \ell_j \in \mathcal{L}(\mathcal{S},\mathcal{S}),\ 1 \le j \le k\}$ in $L^2_{\mathcal{S}}(P)$.

That proposition applies to the above examples. However the application is less explicit since the λ_j's are in general unbounded linear operators which are difficult to specify.

We finally study the structure of $(X_n \otimes X_n')$ where (X_n') is a $ARH'(1)$ process (H' a real separable Hilbert space) associated with ρ' and (ε_n') and which satisfies the same conditions as (X_n). We have the following result

Proposition 3.3. *If $(X_n') \perp\!\!\!\perp (X_n)$ and (ε_n), (ε_n') are martingale differences, then*

$$X_n \otimes X_n' = S(X_{n-1} \otimes X_{n-1}') + G_n \tag{3.12}$$

where S is defined by

$$S(s) = \rho' s \rho^*, \quad s \in \mathcal{S}(H, H') \tag{3.13}$$

and (G_n) is the innovation of $(X_n \otimes X_n')$ and a martingale difference with respect to $\left(\sigma(X_t \otimes X_t',\ t \le n)\right)$.

Clearly this result can be extended to the tensorial product of several Hilbertian autoregressive processes of order 1.

4. Tensorial products of *MAH*(1) processes

Consider the $MAH(1)$

$$X_n = \varepsilon_n + a(\varepsilon_{n-1}) \tag{4.1}$$

where $a \in \mathcal{L}$ is such that $\|a^{j_0}\|_{\mathcal{L}} < 1$ for some $j_0 \ge 1$ and (ε_n) is a H-white noise.

We intend to study the structure of the processes

$$Z_n^{(k)} = X_{n-k} \otimes X_n - \mathbb{E}(X_{n-k} \otimes X_n), \quad n \in \mathbb{Z}\ (k \ge 0), \tag{4.2}$$

where

$$\mathbb{E}(X_n \otimes X_n) = C_{\varepsilon_0} + aC_{\varepsilon_0}a^*,$$
$$\mathbb{E}(X_{n-1} \otimes X_n) = aC_{\varepsilon_0},$$

and

$$\mathbb{E}(X_{n-k} \otimes X_n) = 0, \quad k \ge 2.$$

Now we have

Proposition 4.1. *If A_1 holds then, for all $k \geq 0$, $(Z_n^{(k)})$ is a MA$\mathcal{S}(1)$:*

$$Z_n^{(k)} = E_n^{(k)} + \lambda_k(E_{n-1}^{(k)}), \quad n \in \mathbb{Z} \tag{4.3}$$

where $(E_n^{(k)})$ is the innovation of $(Z_n^{(k)})$ and $\lambda_k(E_{n-1}^{(k)})$ is the orthogonal projection of $(Z_n^{(k)})$ on $\overline{sp}\{\ell(E_{n-1}^{(k)}), \ell \in \mathcal{L}(\mathcal{S}, \mathcal{S})\}$ in $L_H^2(P)$.

Comparing with Proposition 3.2 we notice that the MA structure is more stable than the AR one. This fact is strengthened by the following results:

Proposition 4.2. 1. *If (ε_n) is a* **strong** *white noise and $\varphi : H \mapsto H'$ (a real separable Hilbert space) is measurable and such that $\mathbb{E}\|\varphi(X_n)\|_{H'}^2 < \infty$ and $\mathbb{E}\varphi(X_n) = 0$, then $(\varphi(X_n))$ is a MAH$'(1)$.*

2. *If (X_n) is a standard MAH(1) whose innovation (ε_n) is a martingale difference, and (X_n') is a H'-valued regular stationary process, independent from (X_n), then $(X_n \otimes X_n')$ is a $H \otimes H'$-valued MA(1) process.*

Example 5 (*Truncated Ornstein–Uhlenbeck Process*). The process

$$b_t = \int_{\{t\}-1}^{t} \mathrm{e}^{-\theta(t-s)} \mathrm{d}W(s), \quad t \in \mathbb{R}\ (\theta > 0, \sigma > 0)$$

where W is a bilateral Wiener process and $\{t\}$ is the smallest integer $\geq t - 1$, admits a *MAH*(1) representation in the same space as the Ornstein–Uhlenbeck process (Example 2), with the same innovation, and with $a = \rho$. Since it is gaussian, Proposition 4.1 applies.

The real case If (X_n) is a real MA(1) it is possible to obtain more precise results; suppose that

$$X_n = \varepsilon_n + a\varepsilon_{n-1} \quad (|a| < 1) \tag{4.4}$$

and set

$$\rho = corr(X_{n-1}, X_n), \qquad R = corr(Z_{n-1}^{(0)}, Z_n^{(0)}),$$
$$\delta_{(4)} = \frac{\mathbb{E}\varepsilon_0^4 - 3\sigma^4}{2\sigma^4} \quad (\text{where } \sigma^2 = \mathbb{E}\varepsilon_0^2)$$
$$\alpha(a) = (1 + a^4)/(1 + a^2)^2$$

then we have the following statement

Proposition 4.3. *If (ε_n) satisfies A_1, $(Z_n^{(0)})$ is a MA(1):*

$$Z_n^{(0)} = E_n + AE_{n-1} \quad (|A| < 1) \tag{4.5}$$

where (E_n) is the innovation and A is such that

$$R = \rho^2 \frac{1 + \delta_{(4)}}{1 + \alpha(a)\delta_{(4)}} = \frac{A}{1 + A^2}. \tag{4.6}$$

It follows that $0 \leq R < \frac{a^2}{1+a^4}$ and, if $\delta_{(4)} = 0$, $R = \rho^2$ and A satisfies the relation

$$\frac{A}{1 + A^2} = \left(\frac{a}{1 + a^2}\right)^2 \tag{4.7}$$

thus $0 \leq R < \frac{1}{4}$ and

$$A = \frac{1 - \sqrt{1 - 4R^2}}{4R} \quad (R \neq 0), \tag{4.8}$$

in particular this value of A holds if (X_n) is gaussian.

5. Standard tensorial MA processes

To know if the MA processes obtained in the previous section are standard is somewhat difficult. We now indicate some special cases where that property holds.

First if H (and H') are finite-dimensional the obtained MA are, of course, standard.

The second case occurs if the operator a is **nilpotent**, that is $a^p = 0$ for some $p \geq 2$.

Proposition 5.1. *Let (X_n) be a standard MAH(1) associated with a, (ε_n) and such that A_1 holds. Then, if a is nilpotent, the process*

$$Y_n^{(p)} = a^{p-2}(X_n) \otimes a^{p-2}(X_n) - \mathbb{E}(a^{p-2}(X_n) \otimes a^{p-2}(X_n)), \quad n \in \mathbb{Z} \tag{5.1}$$

is a standard $MA\mathcal{S}(p-1)$ (possibly degenerated):

$$Y_n^{(p)} = \sum_{j=0}^{p-1} A^j(E_{n-p+1,p}) \tag{5.2}$$

where $(E_{n,p},\ n \in \mathbb{Z})$ is the innovation and

$$A(s) = asa^*, \quad s \in \mathcal{S}. \tag{5.3}$$

In particular, if $p = 2$, $(Y_n^{(2)})$ is a standard $MA\mathcal{S}(1)$:

$$X_n \otimes X_n - \mathbb{E}(X_n \otimes X_n) = E_{n,2} + A(E_{n-1,2}).$$

Finally we consider a more useful situation.

Proposition 5.2. *Suppose that (X_n) is a MAH(1) associated with a gaussian innovation (ε_n) and the operator*

$$a = \sum_{j\geq 1} a_j e_j \otimes e_j \tag{5.4}$$

with $1 > |a_1| \geq |a_2| \geq \cdots$ where the e_j's are an orthonormal system of eigenvectors of C_{ε_0}. Then $Z_n^{(0)}$ is standard:

$$Z_n^{(0)} = E_n + A(E_{n-1}) \tag{5.5}$$

where

$$A = \sum_{i,j\geq 1} A_{i,j}(e_i \otimes e_j) \otimes (e_i \otimes e_j) \tag{5.6}$$

with

$$\frac{A_{i,j}}{1 + A_{i,j}^2} = \frac{a_i}{1 + a_i^2} \cdot \frac{a_j}{1 + a_j^2}; \quad i \geq 1, j \geq 1, \tag{5.7}$$

in particular, if a is nuclear, A is nuclear.

We think that the general problem of standardness for tensorial products of standard processes is open.

6. Applications

In this section, we give some brief ideas concerning the possible applications of the above results.

Let $(X_n,\ n \in \mathbb{Z})$ be an ARH(1) process satisfying assumptions in Proposition 3.2. In order to estimate the autocovariance $\Gamma_k = \mathbb{E}(X_0 \otimes X_k)$, $k \geq 0$ one may use the empirical autocovariance

$$\Gamma_{k,n} = \frac{1}{n-k} \sum_{j=1}^{n-k} X_j \otimes X_{j+k},$$

then using the fact that $(Z_n^{(k)})$ is a $ARMA\mathcal{S}(1, k)$ with a standard autoregressive part, one may show that, as $n \to \infty$,

$$n\mathbb{E}\|\Gamma_{k,n} - \Gamma_k\|_{\mathcal{S}}^2 \to c_k,$$

where c_k is an explicit constant.

Concerning MAH(1) processes, if (X_n) satisfies the conditions in Proposition 5.2, it follows that $(Z_n^{(0)})$ is a standard $MA\mathcal{S}(1)$ process:

$$Z_n^0 = E_n + A(E_{n-1}),$$

consequently it is possible to estimate A by using an estimator A_n, based on $X_1 \otimes X_1, \ldots, X_n \otimes X_n$, and defined similarly as in Bosq and Turbillon [12].

Proofs of the above results are beyond the scope of this paper.

7. Proofs and lemmas

The following lemmas will be used in the proofs:

Lemma 7.1. *Let U and V be square integrable real random variables and let Y and Z be $\mathbb{R}^D$-valued random vectors, where D is a countable set. Then, if $(U, Y) \perp (V, Z)$ and $\mathbb{E}(U|Y) = 0$ or $\mathbb{E}(V|Z) = 0$, we have*

$$\mathbb{E}(UV|Y, Z) = 0.$$

The proof is straightforward and therefore omitted.

Lemma 7.2. *Let (X_n) be a stationary regular H-valued process with autocovariance (C_h). Then, (X_n) is a (non-standard) MAH(q) if and only if $C_q \neq 0$ and $C_h = 0$, $|h| > q$.*

This lemma is an extension of Proposition 3.2.1 p. 89 in [13]. The proof appears in [5] and [9].

Proof of Proposition 3.1. Clearly (3.1) and (3.2) yield (3.3)–(3.5). Concerning (3.6) it comes from the relation

$$R^j(s) = \rho^j s \rho^{*j}, \quad s \in \mathcal{S}.$$

Now A_1 entails

$$\mathbb{E}^{\mathcal{B}_{n-1}}\left[(X_{n-1} \otimes \varepsilon_n)\rho^*\right] = \mathbb{E}^{\mathcal{B}_{n-1}}\left[\rho(\varepsilon_n) \otimes X_{n-1}\right] = 0,$$

therefore

$$\mathbb{E}^{\mathcal{B}_{n-1}}(E_n) = 0, \tag{7.1}$$

since a martingale difference with constant covariance operators is a white noise, (3.5) and (7.1) imply that (E_n) is a white noise. Finally the relation

$$Z_n^{(0)} = \sum_{j\geq 0} R^j(E_{n-j}) \quad (\text{in } L_H^2(P)) \tag{7.2}$$

shows that (E_n) is the innovation of $(Z_n^{(0)})$. ■

Proof of Proposition 3.2. Let us set

$$F_n = Z_n^{(k)} - R(Z_{n-1}^{(k)})$$

then we have

$$F_n = \rho(X_{n-k-1}) \otimes \varepsilon_n + \varepsilon_{n-k} \otimes \rho(X_{n-1}) + \varepsilon_{n-k} \otimes \varepsilon_n - \rho^k C_{\varepsilon_0}.$$

Now, noting that

$$X_{n-1} = \sum_{j=0}^{k-1} \rho^j(\varepsilon_{n-1-j}) + \rho^k(X_{n-k-1}) \tag{7.3}$$

one obtains

$$\begin{aligned}\mathbb{E}^{\mathcal{B}_{n-k-1}}(F_n) \;=\; &\rho(X_{n-k-1}) \otimes \mathbb{E}^{\mathcal{B}_{n-k-1}}(\varepsilon_n) + \mathbb{E}^{\mathcal{B}_{n-k-1}}(\varepsilon_{n-k} \otimes \varepsilon_n - \rho^k C_{\varepsilon_0}) \\ &+ \rho(\mathbb{E}^{\mathcal{B}_{n-k-1}}(\varepsilon_{n-k}) \otimes \rho^k(X_{n-k-1})) + \rho \sum_{j=0}^{k-1} \rho^j \mathbb{E}^{\mathcal{B}_{n-k-1}}(\varepsilon_{n-k} \otimes \varepsilon_{n-1-j}),\end{aligned}$$

and A_1 yields

$$\mathbb{E}^{\mathcal{B}_{n-k-1}}(\varepsilon_{n-k}) = \mathbb{E}^{\mathcal{B}_{n-k-1}}(\varepsilon_n) = 0 \tag{7.4}$$

and

$$\begin{aligned}\mathbb{E}^{\mathcal{B}_{n-k-1}}(\varepsilon_{n-k} \otimes \varepsilon_{n-1-j}) &= \mathbb{E}^{\mathcal{B}_{n-k-1}}\mathbb{E}^{\mathcal{B}_{n-k}}(\varepsilon_{n-k} \otimes \varepsilon_{n-1-j}) \\ &= \mathbb{E}^{\mathcal{B}_{n-k-1}}\left(\varepsilon_{n-k} \otimes \mathbb{E}^{\mathcal{B}_{n-k}}(\varepsilon_{n-1-j})\right) \\ &= \begin{cases} 0 & \text{if } j < k-1, \\ C_{\varepsilon_0} & \text{if } j = k-1. \end{cases}\end{aligned}$$

Collecting these results we get

$$\mathbb{E}^{\mathcal{B}_{n-k-1}}(F_n) = 0 \tag{7.5}$$

hence

$$C_{F_m,F_n} = 0 \quad \text{if } |m-n| > k, \tag{7.6}$$

and it is easy to verify that (F_n) is stationary. If (F_n) is regular we may apply Lemma 2 for obtaining (3.11). If it is not regular the same result holds with E_n degenerated since (E_n) is the innovation of (F_n).

Proof of Proposition 3.3. (3.12) and (3.13) follow from the definitions of (X_n) and (X'_n) and one has

$$G_n = \rho(X_{n-1}) \otimes \varepsilon'_n + \varepsilon_n \otimes \rho'(X'_{n-1}) + \varepsilon_n \otimes \varepsilon'_n, \tag{7.7}$$

then if $x \in H$ and $y \in H'$ we may write

$$\langle G_n, x \otimes y\rangle_{\mathcal{S}(H,H')} = \langle \rho(X_{n-1}), x\rangle_H \left\langle \varepsilon'_n, y\right\rangle_{H'} + \langle \varepsilon_n, x\rangle_H \left\langle \rho'(X'_{n-1}), y\right\rangle_{H'} + \langle \varepsilon_n, x\rangle_H \left\langle \varepsilon'_n, y\right\rangle_{H'}.$$

Taking the conditional expectation with respect to the σ-algebra $\mathcal{C}_{n-1} = \sigma(\varepsilon_t, \varepsilon'_t, t \le n-1)$ and applying Lemma 7.1, we obtain

$$\mathbb{E}^{\mathcal{C}_{n-1}}\left(\langle G_n, x \otimes y\rangle_{\mathcal{S}(H,H')}\right) = 0; \quad x \in H, y \in H'$$

thus

$$\mathbb{E}^{\mathcal{C}_{n-1}}(G_n) = 0. \tag{7.8}$$

(7.7) and (7.8) imply that (G_n) is a $\mathcal{S}(H,H')$-white noise. Now, since

$$\|S^j\|_{\mathcal{L}(\mathcal{S}(H,H'),\mathcal{S}(H,H'))} \le \|\rho^j\|_{\mathcal{L}(H,H)} \|\rho'^j\|_{\mathcal{L}(H',H')}, \quad j \ge 1$$

we have

$$\sum_{j\ge 0} \|S^j\|_{\mathcal{L}(\mathcal{S}(H,H'),\mathcal{S}(H,H'))} < \infty,$$

therefore

$$X_n \otimes X'_n = \sum_{j\ge 0} S^j(G_{n-j})$$

and (G_n) is the innovation of $(X_n \otimes X'_n)$.

Proof of Proposition 4.1. Consider the sub-σ-algebra of $\mathcal{B}_n$ defined as

$$\mathcal{C}_n = \sigma(X_{t-k} \otimes X_t, t \le n)$$

and note that $\mathbb{E}^{\mathcal{B}_n}(Y) = 0$ entails $\mathbb{E}^{\mathcal{C}_n}(Y) = 0$.

- If $k = 0$ we have

$$\begin{aligned}\mathbb{E}^{\mathcal{B}_{n-2}}(X_n \otimes X_n) &= \mathbb{E}^{\mathcal{B}_{n-2}}(\varepsilon_n \otimes \varepsilon_n) + \mathbb{E}^{\mathcal{B}_{n-2}}(\varepsilon_n \otimes a(\varepsilon_{n-1})) \\ &\quad + \mathbb{E}^{\mathcal{B}_{n-2}}(a(\varepsilon_{n-1}) \otimes \varepsilon_n) + a\mathbb{E}^{\mathcal{B}_{n-2}}(\varepsilon_{n-1} \otimes \varepsilon_{n-1})a^*\end{aligned}$$

  and A_1 gives

$$\begin{aligned}\mathbb{E}^{\mathcal{B}_{n-2}}(\varepsilon_n \otimes a(\varepsilon_{n-1})) &= \mathbb{E}^{\mathcal{B}_{n-2}}\mathbb{E}^{\mathcal{B}_{n-1}}(\varepsilon_n \otimes a(\varepsilon_{n-1})) \\ &= \mathbb{E}^{\mathcal{B}_{n-2}}\left[\mathbb{E}^{\mathcal{B}_{n-1}}(\varepsilon_n) \otimes a(\varepsilon_{n-1})\right] = 0,\end{aligned}$$

  similarly

$$\mathbb{E}^{\mathcal{B}_{n-2}}(a(\varepsilon_{n-1}) \otimes \varepsilon_n) = 0,$$

  and

$$\mathbb{E}^{\mathcal{B}_{n-2}}(\varepsilon_{n-1} \otimes \varepsilon_{n-1}) = C_{\varepsilon_0},$$

  hence

$$\mathbb{E}^{\mathcal{B}_{n-2}}(X_n \otimes X_n) = C_{\varepsilon_0} + aC_{\varepsilon_0}a^*$$

  thus

$$\mathbb{E}^{\mathcal{B}_{n-2}}(Z_n^{(0)}) = 0,$$

  noting that $(Z_n^{(0)})$ is stationary and applying Lemma 7.2 one obtains (4.3).

- If $k = 1$, we have, for each $p \geq 2$,

$$\mathbb{E}\left[(X_{n-p-1} \otimes X_{n-p}) \otimes (X_{n-1} \otimes X_n)\right] = \mathbb{E}(X_{n-p-1} \otimes X_{n-p}) \otimes \mathbb{E}^{\mathcal{B}_{n-p}}(X_{n-1} \otimes X_n),$$

but

$$\mathbb{E}^{\mathcal{B}_{n-p}}(X_{n-1} \otimes X_n) = \mathbb{E}^{\mathcal{B}_{n-p}}\left[(\varepsilon_{n-1} + a(\varepsilon_{n-2})) \otimes (\varepsilon_n + a(\varepsilon_{n-1}))\right] = aC_{\varepsilon_0}$$

hence

$$\mathbb{E}(Z_{n-p}^{(1)} \otimes Z_n^{(1)}) = 0, \quad p \geq 2$$

and Lemma 7.2 gives (4.3).

- If $k \geq 2$, we have

$$Z_n^{(k)} = X_{n-k} \otimes X_n$$

and, for $p \geq 2$,

$$\mathbb{E}^{\mathcal{B}_{n-p}}(X_{n-k} \otimes X_n) = \mathbb{E}^{\mathcal{B}_{n-p}}\mathbb{E}^{\mathcal{B}_{n-k}}(X_{n-k} \otimes X_n) = \mathbb{E}^{\mathcal{B}_{n-p}}(X_{n-k} \otimes \mathbb{E}^{\mathcal{B}_{n-k}}(X_n)) = 0,$$

and Lemma 7.2 gives (4.3).

Proof of Proposition 4.2. 1. Since (ε_n) is a strong white noise, $(\varphi(X_n))$ is strictly stationary and $\varphi(X_n) \perp\!\!\!\perp \varphi(X_{n-k}), k \geq 2$, hence the result from Lemma 7.2.

2. First, $(X_n \otimes X_n')$ is clearly a zero-mean $\mathcal{S}(H, H')$-valued stationary process. Now consider the σ-algebra

$$\mathcal{C}_n = \sigma(\varepsilon_t, \varepsilon_t', t \leq n) = \sigma(\langle \varepsilon_t, e_j \rangle, \langle \varepsilon_t', e_j' \rangle; t \leq n, j \geq 1)$$

where (ε_n') is the innovation of (X_n'), and (e_j) (resp. (e_j')) is an orthonormal basis of H (resp. H').

If $k \geq 2$, we have, for each $x \in H, y \in H'$,

$$\mathbb{E}^{\mathcal{C}_{n-k}}(\langle X_n \otimes X_n', x \otimes y \rangle_{\mathcal{S}(H,H')}) = \mathbb{E}^{\mathcal{C}_{n-k}}(\langle X_n, x \rangle_H \langle X_n', y \rangle_{H'}).$$

Applying Lemma 7.1 with $U = \langle X_n, x \rangle_H$, $V = \langle X_n', y \rangle_{H'}$, $Y = (\langle \varepsilon_t, e_j \rangle, t \leq n-k, j \geq 1)$ and $Z = (\langle \varepsilon_t', e_j' \rangle, t \leq n-k, j \geq 1)$, one obtains

$$\mathbb{E}^{\mathcal{C}_{n-k}}(\langle X_n \otimes X_n', x \otimes y \rangle_{\mathcal{S}(H,H')}) = 0; \quad x \in H, y \in H'$$

and the result follows from Lemma 7.2.

Proof of Proposition 4.3. (4.5) is a particular case of (4.3). Now let $(C_h, h \geq 0)$ be the autocovariance of $(Z_n^{(0)})$; tedious but simple calculations give

$$C_0 = (1 + a^4)(\mathbb{E}\varepsilon_0^4 - \sigma^4) + 4a^2\sigma^2,$$
$$C_1 = a^2(\mathbb{E}\varepsilon_0^4 - \sigma^4),$$
$$C_k = 0, \quad k > 1.$$

Now, if $C_0 = 0$ one obtains $\mathbb{E}\varepsilon_0^4 = \sigma^4$ and $a = 0$, thus $\rho = \frac{a}{1+a^2} = 0$ and (4.6) holds with $A = R = 0$.

If $C_0 \neq 0$, we have

$$R = \frac{C_1}{C_0} = \frac{a^2(\mathbb{E}\varepsilon_0^4 - \sigma^4)}{(1 + a^4)(\mathbb{E}\varepsilon_0^4 - \sigma^4) + 4a^2\sigma^2} \geq 0$$

which implies

$$R = \rho^2 \frac{1 + \delta_{(4)}}{1 + \alpha(a)\delta_{(4)}}$$

and since $R = \frac{A}{1+A^2}$, (4.6) follows.

On the other hand since the cumulant $\delta_{(4)}$ belongs to $[-1, \infty[$ one may study the variances of R in function of $\delta_{(4)}$ for a fixed, for obtaining $\lim_{\delta_{(4)} \uparrow \infty} R(\delta_{(4)}) = \frac{a}{1+a^4}$, hence $0 \leq R < \frac{a^2}{1+a^4}$.

Finally, if $\delta_{(4)} = 0$, (4.6) gives $R = \rho^2$ and (4.7); since $0 \leq A \leq 1$, (4.8) follows.

Proof of Proposition 5.1. From

$$X_n = \varepsilon_n + a(\varepsilon_{n-1})$$

we deduce that

$$a^{p-1}(X_n) = a^{p-1}(\varepsilon_n),$$

therefore

$$a^{p-2}(X_n) = a^{p-2}(\varepsilon_n) + a^{p-1}(\varepsilon_{n-1})$$
$$= a^{p-2}(\varepsilon_n) + a^{p-1}(X_{n-1}),$$

thus

$$a^{p-2}(X_n) = a[a^{p-2}(X_{n-1})] + a^{p-2}(\varepsilon_n),$$

and it is easy to verify that $(a^{p-2}(\varepsilon_n))$ is a H-white noise (possibly degenerated) which satisfies A_1. Hence $(a^{p-2}(X_n))$ is a $ARH(1)$ associated with the operator $\rho = a$ and, from Proposition 3.1, it follows that $(Y_n^{(p)})$ is a standard $AR\mathcal{S}(1)$ associated with the operator A defined by

$$A(s) = asa^*, \quad s \in \mathcal{S}.$$

Let $(E_{n,p})$ be the innovation of $(Y_n^{(p)})$, we have

$$Y_n^{(p)} = \sum_{j=0}^{\infty} A^j(E_{n-j,p})$$

and since

$$A^p(s) = a^p s a^{*^p} = 0, \quad s \in \mathcal{S},$$

(5.2) follows.

The particular case $p = 2$ is clear.

Proof of Proposition 5.2.

(1) **Preliminaries**. Let us set

$$Z_{nij} = \langle X_n, e_i\rangle\langle X_n, e_j\rangle - \langle C_{X_0}(e_i), e_j\rangle; \quad i, j \geq 1,\ n \in \mathbb{Z};$$

then $(Z_{nij}; i, j \geq 1)$ is the matricial representation of $Z_n^{(0)} = X_n \otimes X_n - C_{X_0}$ in the orthonormal basis of $\mathcal{S}(e_i \otimes e_j; i, j \geq 1)$.

On the other hand we shall use the following formula [14]: let (N_1, N_2, N_3, N_4) be a zero-mean gaussian vector in $\mathbb{R}^4$, then

$$\mathbb{E}(N_1N_2N_3N_4) = \mathbb{E}(N_1N_2)\mathbb{E}(N_3N_4) + \mathbb{E}(N_1N_3)\mathbb{E}(N_2N_4) + \mathbb{E}(N_1N_4)\mathbb{E}(N_2N_3). \tag{7.9}$$

Finally, note that, since

$$C_{X_0} = C_{\varepsilon_0} + aC_{\varepsilon_0}a^*$$

we have

$$C_{X_0} = \sum_{j\geq 1}(1 + a_j^2)c_j e_j \otimes e_j := \sum_{j\geq 1} \lambda_j e_j \otimes e_j$$

where (c_j) is the sequence of the eigenvalues of C_{ε_0}; and, using $D = aC_{\varepsilon_0}$ (where $D = C_{X_0,X_1}$) we get

$$D = D^* = \sum_{j\geq 1} a_j c_j e_j \otimes e_j;$$

of course one has, $\sum_j |\lambda_j| < \infty$ and $\sum_j |a_j c_j| < \infty$.

(2) **Orthogonality of the** Z_{nij}'s. Suppose that $(i', j') \neq (i, j)$ and consider the random variables Z_{nij} and $Z_{mi'j'}$:

- *If* $|m - n| > 1$, one has $X_n \perp\!\!\!\perp X_m$, hence $Z_{nij} \perp Z_{mi'j'}$. The result remains valid if $i' = i$ and $j' = j$.
- *If* $|m - n| = 1$, we put $X_{nj} = \langle X_n, e_j\rangle$ and $\varepsilon_{nj} = \langle \varepsilon_n, e_j\rangle$; $j \geq 1$, $n \in \mathbb{Z}$, and we note that

$$\mathbb{E}(X_{ni}X_{n+1i'}) = \mathbb{E}[(\varepsilon_{ni} + a_i\varepsilon_{n-1i})(\varepsilon_{n+1i'} + a_{i'}\varepsilon_{ni'})]$$
$$= a_{i'}\mathbb{E}[\varepsilon_{ni}\varepsilon_{ni'}]$$
$$= a_{i'}c_i\delta_{ii'},$$

similarly

$$\mathbb{E}(X_{nj}X_{n+1j'}) = a_{j'}c_j\delta_{jj'}.$$

Now (7.9) gives

$$\mathbb{E}(X_{ni}X_{nj}X_{n+1i'}X_{n+1j'}) - \lambda_i\lambda_{i'}\delta_{ij}\delta_{i'j'} = \lambda_i\lambda_{i'}\delta_{ij}\delta_{i'j'} + a_{i'}c_i\delta_{ii'} + a_{j'}c_j\delta_{jj'} + a_{j'}c_i\delta_{ij'}a_{i'}a_j\delta_{ji'} - \lambda_i\lambda_{i'}\delta_{ij}\delta_{i'j'}$$
$$= (a_{i'}c_ia_{j'}c_j)\delta_{ii'}\delta_{jj'} + (a_{j'}c_ia_{i'}c_j)\delta_{ij'}\delta_{ji'}.$$

Now,

$$(i,j) \neq (i',j') \Leftrightarrow \delta_{ii}\delta_{jj'} = 0$$

and

$$i \neq j' \quad \text{or} \quad j \neq i' \Leftrightarrow \delta_{ij'}\delta_{ji'} = 0.$$

On the other hand, noting that $Z_{nij} = Z_{nji}$ we may suppose that $i \leq j$ and $i' \leq j'$. Then $i = j'$ **and** $j = i'$ implies

$$j' = i \leq j = i' \leq j'$$

that is $i = i' = j = j'$. Collecting the above remarks we get

$$Z_{nij} \perp Z_{n+1\quad i'j'}; (i,j) \neq (i',j').$$

- If $m = n$, we have to study the orthogonality of Z_{nij} and $Z_{ni'j'}$ for $i \leq j, i' \leq j'$ and $(i,j) \neq (i',j')$. Note that the vector $(X_{ni}, X_{ni'}, X_{nj}, X_{nj'})$ is gaussian.
- If i, i', j, j' are all distinct, the components of the above vector are independent, thus

$$\mathbb{E}(X_{ni}X_{ni'}X_{nj}X_{nj'}) = 0$$

and since

$$\mathbb{E}(Z_{nij}) = \mathbb{E}(Z_{ni'j'}) = 0$$

it follows that

$$Z_{nij} \perp Z_{ni'j'}$$

- If $i = i'$, then $j \neq j'$ and we have the following cases:
  * j and $j' \neq i$, then $\mathbb{E}(X_{ni}^2X_{nj}X_{nj'}) = \mathbb{E}(X_{ni}^2)\mathbb{E}(X_{nj})\mathbb{E}(X_{nj'}) = 0$ hence (7.9) since $\mathbb{E}(Z_{nij}) = 0$.
  * If $j = i'$, then $\mathbb{E}(X_{ni}^3X_{nj'}) = 0$.
  * If $j' = i = i'$, then $\mathbb{E}(X_{ni}^3X_{nj}) = 0$.
  * Similarly we have the cases $j = j', i \neq i'$.
- If $i = j \neq i' = j'$, then $Z_{nii} = \langle X_n, e_i\rangle^2 - \lambda_i$ and $Z_{ni'i'} = \langle X_n, e_{i'}\rangle^2 - \lambda_{i'}$, and since $\langle X_n, e_i\rangle^2 \perp\!\!\!\perp \langle X_n, e_{i'}\rangle^2$ it follows that $Z_{nii} \perp Z_{ni'i'}$.

Finally $i = j', i' = j$ gives $Z_{nij} = Z_{nii'} = Z_{ni'j'} = Z_{ni'i}$. It follows that (7.9) holds for $(i,j) \neq (i',j')$.

(3) **Defining a white noise.** Using Proposition 4.1 and (7.8) we see that (Z_{nij}) is a real MA(1) for each (i,j):

$$Z_{nij} = E_{nij} + A_{ij}E_{n-1ij}, \quad n \in \mathbb{Z} \quad (i,j \geq 1) \tag{7.10}$$

where (E_{nij}) is the innovation of (Z_{nij}) and $|A_{ij}| < 1$.

Now since

$$E_{nij} = \sum_{k \geq 0}(-A_{ij})^k Z_{n-kij}, \quad n \in \mathbb{Z}; i,j \geq 1$$

it follows that

$$\mathbb{E}(E_{nij}E_{mi'j'}) = \sum_{k \geq 0, k' \geq 0} (-A_{ij})^k(-A_{i'j'})^{k'} E(Z_{n-kij}Z_{m-k'i'j'})$$

then, from the second part of the proof,

$$E_{nij} \perp E_{mi'j'}(i',j') \neq (i,j); \quad n,m \in \mathbb{Z}$$

and, if $(i',j') = (i,j)$ we have

$$E_{nij} \perp E_{mij} \quad (m \neq n)$$

since (E_{nij}) is a white noise.

Consequently $E_n = (E_{nij}, i \geq 1, j \geq 1), n \in \mathbb{Z}$ is a $\mathcal{S}$-valued white noise.

Now using a method similar to the proof of Proposition 4.3 it is easy to obtain the relations

$$\frac{A_{ij}}{1 + A_{ij}^2} = \frac{a_i}{1 + a_i^2} \cdot \frac{a_j}{1 + a_j^2}, ; \quad i \geq 1, j \geq 1.$$

Thus we may defined the bounded operator

$$A = \sum_{i,j} A_{ij}(e_i \otimes e_j) \otimes (e_i \otimes e_j)$$

and we have

$$Z_n^{(0)} = E_n + A(E_{n-1}). \tag{7.11}$$

Finally, since $\|A\|_{\mathcal{L}(\mathcal{S},\mathcal{S})} < 1$ we get

$$E_n = \sum_{k\geq 0} (-A)^k Z_{n-k}$$

thus (E_n) is the innovation of (Z_n). ■

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References

[1] U. Grenander, Abstract Inference, Wiley, New York, 1981.
[2] D. Bosq, Linear Processes in Function Spaces, Theory and Applications, in: Lecture Notes in Statistics, vol. 149, Springer-Verlag, New York, 2000.
[3] F. Ferraty, P. Vieu, Theory and practice, in: Nonparametric Functional Data Analysis, in: Springer Series in Statistics, Springer, New York, 2006.
[4] D. Bosq, Processus linéaires vectoriels et prédiction, C. R. Math. Acad. Sci. Paris 337 (2) (2003) 115–118.
[5] D. Bosq, General linear processes in Hilbert spaces and prediction, J. Statist. Plann. Inference 137 (3) (2007) 879–894.
[6] C.W.J. Granger, P. Newbold, Forecasting transformed series, J. Roy. Statist. Soc. Ser. B 38 (2) (1976) 189–203.
[7] P.C.B. Phillips, V. Solo, Asymptotics for linear processes, Ann. Statist. 20 (2) (1992) 971–1001.
[8] I. Choi, M. Taniguchi, Prediction problems for square-transformed stationary processes, Stat. Inference Stoch. Process. 6 (1) (2003) 43–64.
[9] D. Bosq, D. Blanke, Inference and Prediction in Large Dimensions, in: Wiley Series in Probability and Statistics, John Wiley & Sons Ltd., Chichester, 2007.
[10] N.I. Akhiezer, I.M. Glazman, Theory of linear operators in Hilbert space. Vol. I, Frederick Ungar Publishing Co., New York, 1961.
[11] R.M. Fortet, Vecteurs, Fonctions et Distributions Aléatoires dans les Espaces de Hilbert, Editions Hermès, Paris, 1995.
[12] D. Bosq, C. Turbillon, Estimation par projection de l'opérateur d'une moyenne mobile Hilbertienne, Ann. l'ISUP 53 (1) (2009) 31–46.
[13] P.J. Brockwell, R.A. Davis, Time Series: Theory and Methods, second ed., in: Springer Series in Statistics, Springer-Verlag, New York, 1991.
[14] K. Sobczyk, Stochastic Differential Equations, Kluwer Academic Publishers Group, Dordrecht, 1991.